Appendix A Derivation of Governing Equations

A.1 The Continuity Equation

The derivation of continuity equation based on the fundamental physical principle which is conservation of mass. Consider a fixed finite control volume shown in Figure A.1.

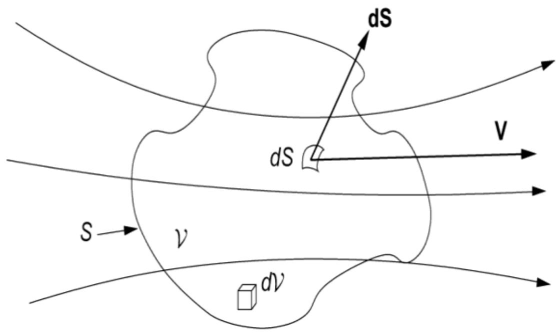

Figure A.1 Finite Control volume fixed in space (Anderson and Wendt, 1995)

From Figure A.1, control volume of arbitrary shape and finite size is considered as the flow model. The surface that bounds this control volume is called as the control surface. The fluid moves through the fixed control volume, flowing across the control surface. According to mass conservation principle that means

Net mass flow out of control volume through surface S	=	Time rate of decrease of mass inside the control volume

or

$$A = B. \tag{A.1}$$

The elemental mass flow across the area dS is given by

$$\rho V_n dS = \rho \boldsymbol{V} \cdot \boldsymbol{dS} \tag{A.2}$$

Note that $\boldsymbol{dS} = \boldsymbol{n} dS$ where $\boldsymbol{n}$ is unit normal vector. By convention, $\boldsymbol{dS}$ always points in a direction out of the control volume. When $\boldsymbol{V}$ also points out of the control volume, the product $\rho\boldsymbol{V} \cdot \boldsymbol{dS}$ is positive. Positive $\rho\boldsymbol{V} \cdot \boldsymbol{dS}$ indicates mass outflow. By summing up the expression on the right-hand side of Equation (A.2) over control surface S and using the theory of limit, the net mass flow out of control volume through surface S is

$$\text{A}=\iint_{\text{S}} \rho\boldsymbol{V} \cdot \boldsymbol{dS}. \tag{A.3}$$

Now consider the right-hand side of Equation (A.1). The mass of an elemental volume $d\mathcal{V}$ is given as $\rho d\mathcal{V}$. Thus, the total mass contained inside the control volume is

$$\iiint_{\mathcal{V}} \rho \, d\mathcal{V}.$$

The time rate of decrease of mass inside control volume $\mathcal{V}$ is given by

$$B = -\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \, d\mathcal{V}. \tag{A.4}$$

Substituting Equation (A.3) and (A.4) into Equation (A.1) gives

$$\iint_{\text{S}} \rho\boldsymbol{V} \cdot \boldsymbol{dS} = -\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \, d\mathcal{V},$$

and by arranging the above equation, it yields

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \, d\mathcal{V} + \iint_{\text{S}} \rho\boldsymbol{V} \cdot \boldsymbol{dS} = 0\,. \tag{A.5}$$

Equation (A.5) is an integral form of continuity equation and it is in conservation form. By applying Gauss divergence theorem to the second term on the LHS of Equation (A.5), this gives

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \, d\mathcal{V} + \iiint_{\mathcal{V}} \nabla \cdot (\rho\boldsymbol{V}) d\mathcal{V} = 0\,. \tag{A.6}$$

The expressions in Equation (A.6) are free from of surface integrals. Since the finite control volume $\mathcal{V}$ is fixed in space, the limits of integration for the integral first term on the LHS of Equation (A.6) are constant,

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho \, d\mathcal{V} = \iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, d\mathcal{V} \, . \tag{A.7}$$

Then, by substituting Equation (A.7) into Equation (A.6), this gives

$$\iiint_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V})\right] d\mathcal{V} = 0 \, . \tag{A.8}$$

Since the finite control volume is arbitrarily drawn in space, the only way for integral in Equation (A.8) to be equal to zero is that the integrand has to be zero at every point within the control volume. Hence, from Equation (A.8), this gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0 \, . \tag{A.9}$$

Equation (A.9) is precisely the partial differential equation of continuity equation. For the two-dimensional flow, vector divergence operator is

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} \, .$$

Note that velocity vector, $\boldsymbol{V} = (u, v)$**.** Equation (A.9) can be expressed as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \, . \tag{A.10}$$

By expanding Equation (A.10) using product rule, it yields

$$\frac{\partial \rho}{\partial t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0 \, ,$$

or in vector form,

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \boldsymbol{V}) + (\boldsymbol{V} \cdot \nabla)\rho = 0 \, . \tag{A.11}$$

By using the definition of material derivative, Equation (A.11) becomes

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{V}) = 0 \,. \tag{A.12}$$

For the case of incompressible flow, density is independent of space and time. Therefore,

$$\frac{D\rho}{Dt} = 0 \,.$$

Hence, from Equation (A.12), this yields

$$\nabla \cdot \boldsymbol{V} = 0 \,,$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \,. \tag{A.13}$$

Equation (A.13) is precisely the governing equation for an incompressible flow. Equation (A.13) can be written in scalar component, that is,

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \,. \tag{A.14}$$

A.2 The Momentum Equation

The fundamental physical principles that required for the derivation of momentum equation namely Newton's second law. As stated by Anderson and Wendt (1995), Newton's second law says that the time rate of change of momentum of a body equals the net force exerted on it. The general statement of this law is given by

$$\frac{d}{dt}(m\boldsymbol{V}) = \boldsymbol{F}, \tag{A.15}$$

where $\boldsymbol{F}$ is the net force exerted on a body. According to Anderson and Wendt (1995), the total forces acting on a body is the sum of body forces and surface forces. These total forces can be written as

$$\boldsymbol{F} = \boldsymbol{F_b} + \boldsymbol{F_s}\,, \tag{A.16}$$

where $\boldsymbol{F_b}$ is the total body force and $\boldsymbol{F_s}$ is the total surface force. Elemental volume $d\mathcal{V}$ consider inside control volume $\mathcal{V}$ shown in Figure A.1, the elemental body force acting on $d\mathcal{V}$ equals to the product of mass and force per unit mass which is $(\rho d\mathcal{V})\boldsymbol{f}$. Hence, the summation of $(\rho d\mathcal{V})\boldsymbol{f}$ over the complete control volume gives

$$\boldsymbol{F_b} = \iiint\limits_{\mathcal{V}} \rho \boldsymbol{f}\, d\mathcal{V}. \tag{A.17}$$

Dissimilar body forces, surface forces act on the boundary of control volume $\boldsymbol{\mathcal{V}}$. Consider an elemental area dS, the elemental surface force is given by

$$\iint\limits_{S} \sigma_n\, dS,$$

where σ_n is the component of stress tensor perpendicular to the surface. By summing the above expression over complete control surface S and using the theory of limit, the total surface force acting on control surface S is

$$\boldsymbol{F_s} = \iint\limits_{S} \sigma_n\, dS = \iint\limits_{S} \boldsymbol{\sigma} \cdot d\boldsymbol{S}, \tag{A.18}$$

where $\boldsymbol{\sigma}$ is the stress tensor vector. Substituting Equation (A.17) and (A.18) into Equation (A.16), the net force is

$$\boldsymbol{F} = \iiint\limits_{\mathcal{V}} \rho \boldsymbol{f}\, d\mathcal{V} + \iint\limits_{S} \boldsymbol{\sigma} \cdot d\boldsymbol{S}. \tag{A.19}$$

Consider the LHS Of Equation (A.15), the elemental momentum flow across the area dS is given by $(\rho\boldsymbol{V}\cdot d\boldsymbol{S})\boldsymbol{V}$. By summing $(\rho\boldsymbol{V}\cdot d\boldsymbol{S})\boldsymbol{V}$ over the complete surface S and using the theory of limit, net rate of flow of momentum over the complete surface S is

$$\mathrm{C}_1 = \iint_S (\rho\boldsymbol{V}\cdot d\boldsymbol{S})\boldsymbol{V}\,. \tag{A.20}$$

If the unsteady flow occurs, there is a fluctuation in momentum. $\rho d\mathcal{V}$ is consider as an elemental mass of fluid and this mass has momentum $\rho\boldsymbol{V}d\boldsymbol{\mathcal{V}}$. By summing $\rho\boldsymbol{V}d\boldsymbol{\mathcal{V}}$ over complete control volume $\mathcal{V}$, the total momentum inside $\mathcal{V}$ is

$$\iiint_{\mathcal{V}} \rho\boldsymbol{V}\,d\mathcal{V}.$$

Since the control volume is fixed in space, the time rate of change of momentum is

$$\mathrm{C}_2 = \frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho\boldsymbol{V}\,d\mathcal{V} = \iiint_{\mathcal{V}} \frac{\partial(\rho\boldsymbol{V})}{\partial t}\,d\mathcal{V}. \tag{A.21}$$

Addition of Equation (A.20) and (A.21) gives

$$\mathrm{C}_1 + C_2 = \iint_S (\rho\boldsymbol{V}\cdot d\boldsymbol{S})\boldsymbol{V} + \iiint_{\mathcal{V}} \frac{\partial(\rho\boldsymbol{V})}{\partial t}\,d\mathcal{V}. \tag{A.22}$$

Hence, substitute Equation (A.19) and (A.22) into Equation (A.15)

$$\iiint_{\mathcal{V}} \frac{\partial(\rho\boldsymbol{V})}{\partial t}\,d\mathcal{V} + \iint_S (\rho\boldsymbol{V}\cdot d\boldsymbol{S})\boldsymbol{V} = \iint_S \boldsymbol{\sigma}\cdot d\boldsymbol{S} + \iiint_{\mathcal{V}} \rho\boldsymbol{f}\,d\mathcal{V}. \tag{A.23}$$

The flow is considered two dimensional. Therefore, Equation (A.23) can be written in component form as

$$\iiint_{\mathcal{V}} \frac{\partial(\rho u_i)}{\partial t}\,d\mathcal{V} + \iint_S \rho u_i u_j n_j dS = \iint_S \sigma_{ij} n_j dS + \iiint_{\mathcal{V}} \rho f_i\,d\mathcal{V}, \tag{A.24}$$

for $i = 1,2$ and $j = 1,2$. Using Gauss divergence theorem,

$$\iint\limits_{S} (\rho u_i u_j n_j) dS = \iiint\limits_{\mathcal{V}} \frac{\partial}{\partial x_j}(\rho u_i u_j) d\mathcal{V},$$

and

$$\iint\limits_{S} \sigma_{ij} n_j dS = \iiint\limits_{\mathcal{V}} \frac{\partial}{\partial x_j}(\sigma_{ij})\, d\mathcal{V}.$$

This leads to

$$\iiint\limits_{\mathcal{V}} \left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial}{\partial x_j}(\sigma_{ij}) - \rho f_i\right] d\mathcal{V} = 0. \tag{A.25}$$

Equation (A.25) is an integral form of momentum equation for $i = 1,2$ and $j = 1,2$. For the integral in Equation (A.25) to be equal to zero, the integrand must be equal to zero at any point within the control volume. Therefore,

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial}{\partial x_j}(\sigma_{ij}) - \rho f_i\,, \tag{A.26}$$

for $i = 1,2$ and $j = 1,2$. Using product rule, expanding Equation (A.26)

$$\rho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \rho}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial(\rho u_j)}{\partial x_j} = \frac{\partial}{\partial x_j}(\sigma_{ij}) + \rho f_i\,.$$

Then, factorise that equation. This equation will become,

$$\rho\frac{\partial u_i}{\partial t} + u_i\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j}\right) + \rho u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(\sigma_{ij}) + \rho f_i\,. \tag{A.27}$$

Since $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} = 0$, Equation (A.27) yields,

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = \frac{\partial}{\partial x_j}(\sigma_{ij}) + \rho f_i\,, \tag{A.28}$$

for $i = 1,2$ and $j = 1,2$.

For an incompressible Newtonian fluid, the shear stress is proportional to the deformation rate that is,

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{A.29}$$

for $i = 1,2$ and $j = 1,2$. By using Equation (A.29), the left-hand side term of Equation (A.28) becomes,

$$\begin{aligned}\frac{\partial}{\partial x_j}(\sigma_{ij}) &= \frac{\partial}{\partial x_j}(-p\delta_{ij} + \tau_{ij}),\\ &= -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j},\\ &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right],\\ &= -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right). \end{aligned} \tag{A.30}$$

Since $\frac{\partial u_j}{\partial x_i} = 0$, Equation (A.30) becomes,

$$\frac{\partial}{\partial x_j}(\sigma_{ij}) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2}.$$

Therefore, Equation (A.28) yields,

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j\frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \rho f_{\boldsymbol{i}}. \tag{A.31}$$

The definition of substantial derivative is applied to the Equation (A.30). Hence, Equation (A.31) becomes,

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i + \rho f_{\boldsymbol{i}}, \tag{A.32}$$

for $i = 1,2$. Equation (A.32) is obtained in a vector form which is,

$$\rho\frac{D\boldsymbol{V}}{Dt} = -\nabla p + \mu\nabla^2\boldsymbol{V} + \rho\boldsymbol{F}. \tag{A.33}$$

For x-direction of momentum equation, the term in the left-hand side of Equation (A.33) becomes,

$$\rho\frac{Du}{Dt}=\frac{\partial(\rho u)}{\partial t}+\boldsymbol{\nabla}\cdot(\rho u\boldsymbol{V}), \tag{A.34}$$

or

$$\rho\frac{Du}{Dt}=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right). \tag{A.35}$$

While, for the y-direction of momentum equation,

$$\rho\frac{Dv}{Dt}=\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right). \tag{A.36}$$

Therefore, Equation (A.33) can be written in both x and y-directions

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\boldsymbol{F_x}\,, \tag{A.37}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\boldsymbol{F_y}\,. \tag{A.38}$$

Body force, $\boldsymbol{F_x}$ and $\boldsymbol{F_y}$ are the magnetic force in FHD where

$$\boldsymbol{F}=\bar{\mu}_0(\mathbf{M}\cdot\nabla)\mathbf{H}\,. \tag{A.39}$$

The magnetic force due to magnetization per unit volume is generally $\bar{\mu}_0(\mathbf{M}\cdot\nabla)\mathbf{H}$ where **M** is the magnetization and **H** is the magnetic field intensity. Assuming that the magnetization is equilibrium, thus **M** and **H** are parallel.

With the help of vector identity, $\bar{\mu}_0(\mathbf{M}\cdot\nabla)\mathbf{H}$ can be rewritten

$$\bar{\mu}_0(\mathbf{M}\cdot\nabla)\mathbf{H}=\bar{\mu}_0[\nabla(\mathbf{M}\cdot\mathbf{H})-\mathbf{H}\cdot\nabla\mathbf{M}-\mathbf{M}\times(\nabla\times\mathbf{H})-\mathbf{H}\times(\nabla\times\mathbf{M})]. \tag{A.40}$$

For constant **M** this simplifies to

$$\bar{\mu}_0(\mathbf{M}\cdot\nabla)\mathbf{H}=\bar{\mu}_0[\nabla(\mathbf{M}\cdot\mathbf{H})-\mathbf{M}\times(\nabla\times\mathbf{H})]. \tag{A.41}$$

When there is no flow of electric current, $\nabla\times\mathbf{H}$ is identically zero, and thus Equation (A.41) yields

$$\bar{\mu}_0(\mathbf{M}\cdot\nabla)\mathbf{H} = \bar{\mu}_0[\nabla(\mathbf{M}\cdot\mathbf{H})]$$
$$= \bar{\mu}_0\mathbf{M}\nabla\mathbf{H} \qquad \text{(A.42)}$$

The scalar component due to FHD, $\boldsymbol{F} = \bar{\mu}_0\bar{M}\nabla\bar{H}$ are given by

$$\boldsymbol{F_x} = \bar{\mu}_0\bar{M}\frac{\partial\bar{H}}{\partial\bar{x}}, \qquad \text{(A.43)}$$

$$\boldsymbol{F_y} = \bar{\mu}_0\bar{M}\frac{\partial\bar{H}}{\partial\bar{y}}. \qquad \text{(A.44)}$$

The case study considers the flow is steady incompressible magnetic fluid flow in two dimensional. Now, the continuity and momentum equations can be written in scalar form in x and y-direction respectively,

$$\frac{\partial\bar{u}}{\partial\bar{x}} + \frac{\partial\bar{v}}{\partial\bar{y}} = 0, \qquad \text{(A.45)}$$

$$\bar{\rho}\left(\bar{u}\frac{\partial\bar{u}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{u}}{\partial\bar{y}}\right) = -\frac{\partial\bar{p}}{\partial\bar{x}} + \bar{\mu}\left(\frac{\partial^2\bar{u}}{\partial\bar{x}^2} + \frac{\partial^2\bar{u}}{\partial\bar{y}^2}\right) + \bar{\mu}_0\bar{M}\frac{\partial\bar{H}}{\partial\bar{x}}, \qquad \text{(A.46)}$$

$$\bar{\rho}\left(\bar{u}\frac{\partial\bar{v}}{\partial\bar{x}} + \bar{v}\frac{\partial\bar{v}}{\partial\bar{y}}\right) = -\frac{\partial\bar{p}}{\partial\bar{y}} + \bar{\mu}\left(\frac{\partial^2\bar{v}}{\partial\bar{x}^2} + \frac{\partial^2\bar{v}}{\partial\bar{y}^2}\right) + \bar{\mu}_0\bar{M}\frac{\partial\bar{H}}{\partial\bar{y}}. \qquad \text{(A.47)}$$

A.3 Non-dimensionalization of Magnitude $\bar{H}$ of the Magnetic Field Intensity

The magnitude $\bar{H}$, of the magnetic field intensity, is given by

$$\bar{H} = \left[\bar{H}_x{}^2 + \bar{H}_y{}^2\right]^{\frac{1}{2}} = \frac{\gamma}{2\pi}\frac{1}{\sqrt{(\bar{x}-\bar{a})^2 + \left(\bar{y}-\bar{b}\right)^2}}. \qquad \text{(A.48)}$$

Equation (A.48) could be nondimensionalised by using the following non-dimensional variables

$$x = \frac{\bar{x}}{\bar{h}}, \qquad y = \frac{\bar{y}}{\bar{h}}, \qquad H = \frac{\bar{H}}{\bar{H}_0}. \tag{A.49}$$

where $\bar{H}_0$ is the magnetic field intensity. Note that $\bar{H}_0 = (\bar{a}, 0)$ substitute this into Equation (A.48) to get

$$\begin{aligned}
\bar{H}_0 = \bar{H}(\bar{a}, 0) &= \frac{\gamma}{2\pi} \frac{1}{\sqrt{(\bar{a} - \bar{a})^2 + \left(0 - \bar{b}\right)^2}} \\
&= \frac{\gamma}{2\pi} \frac{1}{\sqrt{\left(-\bar{b}\right)^2}} \\
&= \frac{\gamma}{2\pi} \frac{1}{\left|-\bar{b}\right|}
\end{aligned} \tag{A.50}$$

γ is the magnetic field strength at the point $\left(\bar{x} = \bar{a}, \bar{y} = \bar{b}\right)$

Apply the dimensionless variables in Equation (A.49) and equation for $\bar{H}_0$ from Equation (A.50) into Equation (A.48) to obtain H. The steps are as shown below

$$\begin{aligned}
\bar{H} &= \frac{\gamma}{2\pi} \frac{1}{\sqrt{(\bar{x} - \bar{a})^2 + \left(\bar{y} - \bar{b}\right)^2}} \\
H\bar{H}_0 &= \frac{\gamma}{2\pi} \frac{1}{\sqrt{\left(x\bar{h} - a\bar{h}\right)^2 + \left(y\bar{h} - b\bar{h}\right)^2}} \\
H\left(\frac{\gamma}{2\pi} \frac{1}{\left|-b\bar{h}\right|}\right) &= \frac{\gamma}{2\pi} \frac{1}{\sqrt{\left(x\bar{h} - a\bar{h}\right)^2 + \left(y\bar{h} - b\bar{h}\right)^2}} \\
H\left(\frac{1}{\bar{h}|-b|}\right) &= \frac{1}{\sqrt{\left(\bar{h}\right)^2}\sqrt{(x - a)^2 + (y - b)^2}} \\
H &= \frac{|b|}{\sqrt{(x - a)^2 + (y - b)^2}}
\end{aligned} \tag{A.51}$$